CHARACTERS OF INTEGRABLE HIGHEST WEIGHT MODULES OVER A QUANTUM GROUP

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ABSTRACT. We show that the Weyl-Kac type character formula holds for the integrable highest weight modules over the quantized enveloping algebra of any symmetrizable Kac-Moody Lie algebra, when the parameter q is not a root of unity.

1. INTRODUCTION

It is well-known that the character of an integrable highest weight module over a symmetrizable Kac-Moody algebra $\mathfrak{g}$ is given by the Weyl-Kac character formula (see Kac [6]). In this paper we consider the corresponding problem for a quantized enveloping algebra (see Kashiwara [7]).

For a field K and $z \in K^{\times}$ which is not a root of 1, we denote by $U_{K,z}(\mathfrak{g})$ the quantized enveloping algebra of $\mathfrak{g}$ over K at $q = z$, namely the specialization of Lusztig's $\mathbb{Z}[q, q^{-1}]$-form via $q \mapsto z$. It is already known that the Weyl-Kac type character formula holds for $U_{K,z}(\mathfrak{g})$ in some cases. When K is of characteristic 0 and z is transcendental, this is due to Lusztig [10]. When $\mathfrak{g}$ is finite-dimensional, this is shown in Andersen, Polo and Wen [1]. When $\mathfrak{g}$ is affine, this is known in certain specific cases (see Chari and Jing [2], Tsuchioka [15]).

We first point out that the problem is closely related to the non-degeneracy of the Drinfeld pairing for $U_{K,z}(\mathfrak{g})$. In fact, assume we could show that the Drinfeld pairing for $U_{K,z}(\mathfrak{g})$ is non-degenerate. Then we can define the quantum Casimir operator. It allows us to apply Kac's argument for Lie algebras in [6] to $U_{K,z}(\mathfrak{g})$, and we obtain the Weyl-Kac type character formula for integrable highest weight modules over $U_{K,z}(\mathfrak{g})$. In particular, we can deduce the Weyl-Kac type character formula in the affine case from the case-by-case calculation of the Drinfeld pairing due to Damiani [3], [4].

The aim of this paper is to give a simple unified proof of the non-degeneracy of the Drinfeld pairing and the Weyl-Kac type character formula for $U_{K,z}(\mathfrak{g})$, where $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra, K is a field not necessarily of characteristic zero, and $z \in K^{\times}$ is not a root of 1. Our argument is as follows. We consider the (possibly)

modified algebra $\overline{U}_{K,z}(\mathfrak{g})$, which is the quotient of $U_{K,z}(\mathfrak{g})$ by the ideal generated by the radical of the Drinfeld pairing. Then the Drinfeld pairing for $U_{K,z}(\mathfrak{g})$ induces a non-degenerate pairing for $\overline{U}_{K,z}(\mathfrak{g})$, by which we can define the quantum Casimir operator for $\overline{U}_{K,z}(\mathfrak{g})$. It allows us to apply Kac's argument for Lie algebras to $\overline{U}_{K,z}(\mathfrak{g})$, and we obtain the Weyl-Kac type character formula for $\overline{U}_{K,z}(\mathfrak{g})$ with modified denominator. In the special case where the highest weight is zero, this gives a formula for the modified denominator. Comparing this with the ordinary denominator formula for Lie algebras, we conclude that the modified denominator coincides with the original denominator for the Lie algebra $\mathfrak{g}$. It implies that the Drinfeld pairing for $U_{K,z}(\mathfrak{g})$ was already non-degenerate. This is the outline of our argument. In applying Kac's argument to the modified algebra, we need to show that the modified denominator is skew invariant with respect to a twisted action of the Weyl group. This is accomplished using certain standard properties of the Drinfeld pairing.

The first draft of this paper contained only results when K is of characteristic zero. Then Masaki Kashiwara pointed out to me that the arguments work for positive characteristic case as well. I would like to thank Masaki Kashiwara for this crucial remark.

2. QUANTIZED ENVELOPING ALGEBRAS

Let $\mathfrak{h}$ be a finite-dimensional vector space over $\mathbb{Q}$, and let $\{h_i\}_{i \in I}$ and $\{\alpha_i\}_{i \in I}$ be linearly independent subsets of $\mathfrak{h}$ and $\mathfrak{h}^*$, respectively such that $(\langle \alpha_j, h_i \rangle)_{i,j \in I}$ is a symmetrizable generalized Cartan matrix. We denote by W the associated Weyl group. It is a subgroup of $GL(\mathfrak{h})$ generated by the involutions s_i $(i \in I)$ defined by $s_i(h) = h - \langle \alpha_i, h \rangle h_i$ for $h \in \mathfrak{h}$. The contragredient action of W on $\mathfrak{h}^*$ is given by $s_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i$ for $i \in I$, $\lambda \in \mathfrak{h}^*$. Set

$$E = \sum_{i \in I} \mathbb{Q}\alpha_i, \qquad Q = \sum_{i \in I} \mathbb{Z}\alpha_i, \qquad Q^+ = \sum_{i \in I} \mathbb{Z}_{\geqq 0}\alpha_i.$$

We can take a symmetric W-invariant bilinear form $(\ ,\ ): E \times E \to \mathbb{Q}$ such that

$$(2.1) \qquad\qquad \frac{(\alpha_i, \alpha_i)}{2} \in \mathbb{Z}_{>0} \qquad (i \in I).$$

For $\lambda \in E$ and $i \in I$ we obtain from $(\lambda, \alpha_i) = (s_i\lambda, s_i\alpha_i)$ that

$$(2.2) \qquad\qquad \langle \lambda, h_i \rangle = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}.$$

In particular we have

$$(\alpha_i, \alpha_j) = \langle \alpha_j, h_i \rangle \frac{(\alpha_i, \alpha_i)}{2} \in \mathbb{Z},$$

and hence $(Q, Q) \subset \mathbb{Z}$. For $i \in I$ set $t_i = \frac{(\alpha_i, \alpha_i)}{2} h_i$, and for $\gamma = \sum_i n_i \alpha_i \in Q$ set $t_\gamma = \sum_i n_i t_i$. By (2.2) we have $(\lambda, \gamma) = \langle \lambda, t_\gamma \rangle$ for $\lambda \in E$, $\gamma \in Q$. We fix a $\mathbb{Z}$-form $\mathfrak{h}_\mathbb{Z}$ of $\mathfrak{h}$ such that

$$(2.3) \qquad \langle \alpha_i, \mathfrak{h}_\mathbb{Z} \rangle \subset \mathbb{Z}, \qquad t_i \in \mathfrak{h}_\mathbb{Z} \qquad (i \in I).$$

We set

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \mathfrak{h}_\mathbb{Z} \rangle \subset \mathbb{Z}\}, \qquad P^+ = \{\lambda \in P \mid \langle \lambda, h_i \rangle \in \mathbb{Z}_{\geqq 0}\}.$$

We fix $\rho \in \mathfrak{h}^*$ such that $\langle \rho, h_i \rangle = 1$ for any $i \in I$, and define a twisted action of W on $\mathfrak{h}^*$ by

$$w \circ \lambda = w(\lambda + \rho) - \rho \qquad (w \in W, \lambda \in \mathfrak{h}^*).$$

This action does not depend on the choice of ρ, and we have $w \circ P = P$ for any $w \in W$.

Denote by $\mathcal{E}$ the set of formal sums $\sum_{\lambda \in P} c_\lambda e(\lambda)$ $(c_\lambda \in \mathbb{Z})$ such that there exist finitely many $\lambda_1, \ldots, \lambda_r \in P$ such that

$$\{\lambda \in P \mid c_\lambda \neq 0\} \subset \bigcup_{k=1}^r (\lambda_k - Q^+).$$

Note that $\mathcal{E}$ is naturally a commutative ring by the multiplication $e(\lambda)e(\mu) = e(\lambda + \mu)$.

Denote by Δ^+ the set of positive roots for the Kac-Moody Lie algebra $\mathfrak{g}$ associated to the generalized Cartan matrix $(\langle \alpha_j, h_i \rangle)_{i,j \in I}$. For $\alpha \in \Delta^+$ let m_α be the dimension of the root space of $\mathfrak{g}$ with weight α. We define an invertible element D of $\mathcal{E}$ by

$$D = \prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{m_\alpha}.$$

For $n \in \mathbb{Z}_{\geqq 0}$ set

$$[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}} \in \mathbb{Z}[x, x^{-1}], \quad [n]!_x = [n]_x [n-1]_x \cdots [1]_x \in \mathbb{Z}[x, x^{-1}].$$

We denote by $\mathbb{F} = \mathbb{Q}(q)$ the field of rational functions in the variable q with coefficients in $\mathbb{Q}$.

The quantized enveloping algebra U associated to $\mathfrak{h}$, $\{h_i\}_{i \in I}$, $\{\alpha_i\}_{i \in I}$, $\mathfrak{h}_\mathbb{Z}$, $(\ ,\ )$ is the associative algebra over $\mathbb{F}$ generated by the elements k_h,

e_i, f_i ($h \in \mathfrak{h}_\mathbb{Z}$, $i \in I$) satisfying the relations

$$(2.4) \qquad k_0 = 1, \qquad k_h k_{h'} = k_{h+h'} \qquad\qquad (h, h' \in \mathfrak{h}_\mathbb{Z}),$$

$$(2.5) \qquad k_h e_i k_{-h} = q_i^{\langle \alpha_i, h \rangle} e_i \qquad\qquad (h \in \mathfrak{h}_\mathbb{Z}, i \in I),$$

$$(2.6) \qquad k_h f_i k_{-h} = q_i^{-\langle \alpha_i, h \rangle} f_i \qquad\qquad (h \in \mathfrak{h}_\mathbb{Z}, i \in I),$$

$$(2.7) \qquad e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \qquad\qquad (i, j \in I),$$

$$(2.8) \qquad \sum_{r+s=1-\langle \alpha_j, h_i \rangle} (-1)^r e_i^{(r)} e_j e_i^{(s)} = 0 \qquad (i, j \in I, \; i \neq j),$$

$$(2.9) \qquad \sum_{r+s=1-\langle \alpha_j, h_i \rangle} (-1)^r f_i^{(r)} f_j f_i^{(s)} = 0 \qquad (i, j \in I, \; i \neq j),$$

where $k_i = k_{t_i}$, $q_i = q^{(\alpha_i, \alpha_i)/2}$ for $i \in I$, and $e_i^{(r)} = \frac{1}{[r]!_{q_i}} e_i^r$, $f_i^{(r)} = \frac{1}{[r]!_{q_i}} f_i^r$ for $i \in I$, $r \in \mathbb{Z}_{\geqq 0}$. For $\gamma \in Q$ we set $k_\gamma = k_{t_\gamma}$.

We have a Hopf algebra structure of U given by

$$(2.10) \qquad \Delta(k_h) = k_h \otimes k_h,$$

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i$$

$$(2.11) \qquad \varepsilon(k_h) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0,$$

$$(2.12) \qquad S(k_h) = k_h^{-1}, \quad S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i$$

for $h \in \mathfrak{h}_\mathbb{Z}, i \in I$. We will sometimes use Sweedler's notation for the coproduct;

$$\Delta(u) = \sum_{(u)} u_{(0)} \otimes u_{(1)} \qquad (u \in U),$$

and the iterated coproduct;

$$\Delta_m(u) = \sum_{(u)_m} u_{(0)} \otimes \cdots \otimes u_{(m)} \qquad (u \in U).$$

We define $\mathbb{F}$-subalgebras U^0, U^+, U^-, $U^{\geqq 0}$, $U^{\leqq 0}$ of U by

$$U^0 = \langle k_h \mid h \in \mathfrak{h}_\mathbb{Z} \rangle, \quad U^+ = \langle e_i \mid i \in I \rangle, \quad U^- = \langle f_i \mid i \in I \rangle,$$

$$U^{\geqq 0} = \langle k_h, e_i \mid h \in \mathfrak{h}_\mathbb{Z}, i \in I \rangle, \quad U^{\leqq 0} = \langle k_h, f_i \mid h \in \mathfrak{h}_\mathbb{Z}, i \in I \rangle.$$

For $\gamma \in Q$ set

$$U_\gamma = \{ u \in U \mid k_h u k_h^{-1} = q^{\langle \gamma, h \rangle} u \; (h \in \mathfrak{h}_\mathbb{Z}) \}, \qquad U_\gamma^\pm = U_\gamma \cap U^\pm.$$

Then we have

$$U^0 = \bigoplus_{h \in \mathfrak{h}_\mathbb{Z}} \mathbb{F} k_h, \qquad U^\pm = \bigoplus_{\gamma \in Q^+} U^\pm_{\pm \gamma}.$$

It is known that the multiplication of U induces isomorphisms

$$U \cong U^+ \otimes U^0 \otimes U^- \cong U^- \otimes U^0 \otimes U^+,$$

$$U^{\geqq 0} \cong U^+ \otimes U^0 \cong U^0 \otimes U^+, \quad U^{\leqq 0} \cong U^- \otimes U^0 \cong U^0 \otimes U^-$$

of vector spaces. It is also known that

$$(2.13) \qquad \sum_{\gamma \in Q^+} \dim U_{-\gamma}^- e(-\gamma) = D^{-1}.$$

For a U-module V and $\lambda \in P$ we set

$$V_\lambda = \{v \in V \mid k_h v = q^{\langle \lambda, h \rangle} v \ (h \in \mathfrak{h}_{\mathbb{Z}})\}.$$

We say that a U-module V is integrable if $V = \bigoplus_{\lambda \in P} V_\lambda$ and for any $v \in V$ and $i \in I$ there exists some $N > 0$ such that $e_i^{(n)} v = f_i^{(n)} v = 0$ for $n \geqq N$.

For $i \in I$ and an integrable U-module V define an operator $T_i : V \to V$ by

$$T_i v = \sum_{-a+b-c=\langle \lambda, h_i \rangle} (-1)^b q_i^{-ac+b} e_i^{(a)} f_i^{(b)} e_i^{(c)} v \qquad (v \in V_\lambda).$$

It is invertible, and satisfies $T_i V_\lambda = V_{s_i \lambda}$ for $\lambda \in P$. There exists a unique algebra automorphism $T_i : U \to U$ such that for any integrable U-module V we have $T_i uv = T_i(u) T_i v$ ($u \in U, v \in V$). Then we have $T_i(U_\gamma) = U_{s_i \gamma}$ for $\gamma \in Q$. The action of T_i on U is given by

$$T_i(k_h) = k_{s_i h}, \quad T_i(e_i) = -f_i k_i, \quad T_i(f_i) = -k_i^{-1} e_i \qquad (h \in \mathfrak{h}_{\mathbb{Z}}),$$

$$T_i(e_j) = \sum_{r+s=-\langle \alpha_j, h_i \rangle} (-1)^r q_i^{-r} e_i^{(s)} e_j e_i^{(r)} \qquad (j \in I, i \neq j),$$

$$T_i(f_j) = \sum_{r+s=-\langle \alpha_j, h_i \rangle} (-1)^r q_i^r f_i^{(r)} f_j f_i^{(s)} \qquad (j \in I, i \neq j)$$

(see [11, Section 37.1]).

The multiplication of U induces

$$(2.14) \qquad U^+ \cong (U^+ \cap T_i(U^+)) \otimes \mathbb{F}[e_i] \cong \mathbb{F}[e_i] \otimes (U^+ \cap T_i^{-1}(U^+)),$$

$$(2.15) \qquad U^- \cong (U^- \cap T_i(U^-)) \otimes \mathbb{F}[f_i] \cong \mathbb{F}[f_i] \otimes (U^- \cap T_i^{-1}(U^-))$$

(see [11, Lemma 38.1.2]). Moreover,

$$(2.16) \qquad \Delta(U^+ \cap T_i(U^+)) \subset U^{\geqq 0} \otimes (U^+ \cap T_i(U^+)),$$

$$(2.17) \qquad \Delta(U^+ \cap T_i^{-1}(U^+)) \subset U^0(U^+ \cap T_i^{-1}(U^+)) \otimes U^+,$$

$$(2.18) \qquad \Delta(U^- \cap T_i(U^-)) \subset (U^- \cap T_i(U^-)) \otimes U^{\leqq 0},$$

$$(2.19) \qquad \Delta(U^- \cap T_i^{-1}(U^-)) \subset U^- \otimes U^0(U^- \cap T_i^{-1}(U^-))$$

(see [14, Lemma 2.8]).

Set

$$\sharp U^0 = \bigoplus_{\gamma \in Q} \mathbb{F} k_\gamma \subset U^0, \qquad \sharp U^{\geqq 0} = \sharp U^0 U^+, \qquad \sharp U^{\leqq 0} = \sharp U^0 U^-.$$

They are Hopf subalgebras of U. The Drinfeld pairing is the bilinear form

$$\tau : \sharp U^{\geqq 0} \times \sharp U^{\leqq 0} \to \mathbb{F}$$

characterized by the following properties:

$$(2.20) \quad \tau(x, y_1 y_2) = (\tau \otimes \tau)(\Delta(x), y_1 \otimes y_2) \quad (x \in {}^\sharp U^{\geqq 0}, y_1, y_2 \in {}^\sharp U^{\leqq 0}),$$

$$(2.21) \quad \tau(x_1 x_2, y) = (\tau \otimes \tau)(x_2 \otimes x_1, \Delta(y)) \quad (x_1, x_2 \in {}^\sharp U^{\geqq 0}, y \in {}^\sharp U^{\leqq 0}),$$

$$(2.22) \quad \tau(k_\gamma, k_\delta) = q^{-(\gamma, \delta)} \qquad\qquad\qquad\qquad (\gamma, \delta \in Q),$$

$$(2.23) \quad \tau(e_i, f_j) = -\delta_{ij}(q_i - q_i^{-1})^{-1} \qquad\qquad\qquad (i, j \in I),$$

$$(2.24) \quad \tau(e_i, k_\gamma) = \tau(k_\gamma, f_i) = 0 \qquad\qquad\qquad (i \in I, \gamma \in Q).$$

It satisfies the following properties:

$$(2.25) \qquad \tau(x k_\gamma, y k_\delta) = \tau(x, y) q^{-(\gamma, \delta)} \qquad (x \in U^+, y \in U^-, \gamma, \delta \in Q),$$

$$(2.26) \qquad \tau(U_\gamma^+, U_{-\delta}^-) = \{0\} \qquad\qquad\qquad (\gamma, \delta \in Q^+, \gamma \neq \delta),$$

$$(2.27) \qquad \tau|_{U_\gamma^+ \times U_{-\gamma}^-} \text{ is non-degenerate} \qquad\qquad (\gamma \in Q^+),$$

$$(2.28) \qquad \tau(Sx, Sy) = \tau(x, y) \qquad\qquad (x \in {}^\sharp U^{\geqq 0}, y \in {}^\sharp U^{\leqq 0}).$$

Moreover, for $x \in {}^\sharp U^{\geqq 0}$, $y \in {}^\sharp U^{\leqq 0}$ we have

$$(2.29) \qquad xy = \sum_{(x)_2, (y)_2} \tau(x_{(0)}, y_{(0)}) \tau(x_{(2)}, Sy_{(2)}) y_{(1)} x_{(1)},$$

$$(2.30) \qquad yx = \sum_{(x)_2, (y)_2} \tau(Sx_{(0)}, y_{(0)}) \tau(x_{(2)}, y_{(2)}) x_{(1)} y_{(1)}.$$

(see [12, Lemma 2.1.2]).

For $i \in I$ we define linear maps

$$r_{i,\pm} : U^\pm \to U^\pm, \qquad r'_{i,\pm} : U^\pm \to U^\pm$$

by

$$\Delta(x) \in r_{i,+}(x) k_i \otimes e_i + \sum_{\delta \in Q^+ \backslash \{\alpha_i\}} U^{\geqq 0} \otimes U_\delta^+ \qquad (x \in U^+),$$

$$\Delta(x) \in e_i k_{\gamma - \alpha_i} \otimes r'_{i,+}(x) + \sum_{\delta \in Q^+ \backslash \{\alpha_i\}} U_\delta^+ U^0 \otimes U^+ \qquad (x \in U_\gamma^+),$$

$$\Delta(y) \in r_{i,-}(y) \otimes f_i k_{-\gamma + \alpha_i} + \sum_{\delta \in Q^+ \backslash \{\alpha_i\}} U^- \otimes U_{-\delta}^- U^0 \qquad (y \in U_{-\gamma}^-),$$

$$\Delta(y) \in f_i \otimes r'_{i,-}(y) k_i^{-1} + \sum_{\delta \in Q^+ \backslash \{\alpha_i\}} U_{-\delta}^- \otimes U^{\leqq 0} \qquad (y \in U^-).$$

We have

(2.31)
$$U^+ \cap T_i(U^+) = \{u \in U^+ \mid \tau(u, U^- f_i) = \{0\}\}$$
$$= \{u \in U^+ \mid r_{i,+}(u) = 0\},$$

(2.32)
$$U^+ \cap T_i^{-1}(U^+) = \{u \in U^+ \mid \tau(u, f_i U^-) = \{0\}\}$$
$$= \{u \in U^+ \mid r'_{i,+}(u) = 0\},$$

(2.33)
$$U^- \cap T_i(U^-) = \{u \in U^- \mid \tau(U^+ e_i, u) = \{0\}\}$$
$$= \{u \in U^- \mid r'_{i,-}(u) = 0\},$$

(2.34)
$$U^- \cap T_i^{-1}(U^-) = \{u \in U^- \mid \tau(e_i U^+, u) = \{0\}\}$$
$$= \{u \in U^- \mid r_{i,-}(u) = 0\}$$

(see [11, Proposition 38.1.6]).

By (2.16), (2.17), (2.18), (2.19), (2.31), (2.32), (2.33), (2.34) we easily obtain

(2.35)
$$\tau(x e_i^m, y f_i^n) = \delta_{mn} \tau(x, y) \frac{q_i^{n(n-1)/2}}{(q_i^{-1} - q_i)^n} [n]!_{q_i}$$
$$(x \in U^+ \cap T_i(U^+), y \in U^- \cap T_i(U^-)),$$

(2.36)
$$\tau(e_i^m x', f_i^n y') = \delta_{mn} \tau(x', y') \frac{q_i^{n(n-1)/2}}{(q_i^{-1} - q_i)^n} [n]!_{q_i}$$
$$(x' \in U^+ \cap T_i^{-1}(U^+), y' \in U^- \cap T_i^{-1}(U^-)).$$

We have also

(2.37)
$$\tau(x, y) = \tau(T_i^{-1}(x), T_i^{-1}(y))$$
$$(x \in U^+ \cap T_i(U^+), y \in U^- \cap T_i(U^-))$$

(see [11, Proposition 38.2.1], [14, Theorem 5.1]).

3. SPECIALIZATION

Let R be a subring of $\mathbb{F} = \mathbb{Q}(q)$ containing $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$. We denote by U_R the R-subalgebra of U generated by k_h, $e_i^{(n)}$, $f_i^{(n)}$ ($h \in \mathfrak{h}_{\mathbb{Z}}, i \in I, n \geqq 0$). It is a Hopf algebra over R.

We define subalgebras U_R^0, U_R^+, U_R^-, $U_R^{\geqq 0}$, $U_R^{\leqq 0}$ of U_R by

$$U_R^0 = U^0 \cap U_R, \quad U_R^{\pm} = U^{\pm} \cap U_R,$$
$$U_R^{\geqq 0} = U^{\geqq 0} \cap U_R, \quad U_R^{\leqq 0} = U^{\leqq 0} \cap U_R.$$

Setting $U_{R, \pm\gamma}^{\pm} = U_{\pm\gamma}^{\pm} \cap U_R$ for $\gamma \in Q^+$ we have

$$U_R^{\pm} = \bigoplus_{\gamma \in Q^+} U_{R, \pm\gamma}^{\pm}.$$

It is known that $U_{R,\pm\gamma}^{\pm}$ is a free R-module of rank $\dim U_{\pm\gamma}^{\pm}$ (see [11, Section 14.2]). Hence we have

$$(3.1) \qquad \sum_{\gamma \in Q^+} \mathrm{rank}_R(U_{R,-\gamma}^-)e(-\gamma) = D^{-1}$$

by (2.13).

The multiplication of U_R induces isomorphisms

$$U_R \cong U_R^+ \otimes U_R^0 \otimes U_R^- \cong U_R^- \otimes U_R^0 \otimes U_R^+,$$

$$U_R^{\geqq 0} \cong U_R^+ \otimes U_R^0 \cong U_R^0 \otimes U_R^+, \quad U_R^{\leqq 0} \cong U_R^- \otimes U_R^0 \cong U_R^0 \otimes U_R^-$$

of R-modules.

For $i \in I$ the algebra automorphisms $T_i^{\pm 1} : U \to U$ preserve U_R.

LEMMA **3.1.** *The multiplication of U_R induces isomorphisms*

$$(3.2) \qquad U_R^+ \cong (U_R^+ \cap T_i(U_R^+)) \otimes_R \left(\bigoplus_{n=0}^{\infty} R e_i^{(n)} \right),$$

$$(3.3) \qquad U_R^+ \cong \left(\bigoplus_{n=0}^{\infty} R e_i^{(n)} \right) \otimes_R (U_R^+ \cap T_i^{-1}(U_R^+)),$$

$$(3.4) \qquad U_R^- \cong (U_R^- \cap T_i(U_R^-)) \otimes_R \left(\bigoplus_{n=0}^{\infty} R f_i^{(n)} \right),$$

$$(3.5) \qquad U_R^- \cong \left(\bigoplus_{n=0}^{\infty} R f_i^{(n)} \right) \otimes_R (U_R^- \cap T_i^{-1}(U_R^-)).$$

PROOF. We only show (3.2). The injectivity of the canonical homomorphism

$$(U_R^+ \cap T_i(U_R^+)) \otimes_R \left(\bigoplus_{n=0}^{\infty} R e_i^{(n)} \right) \to U_R^+$$

is clear. To show the surjectivity it is sufficient to verify that its image is stable under the left multiplication by $e_j^{(n)}$ for any $j \in I$ and $n \geqq 0$. If $j \neq i$, this is clear since $e_j^{(n)} \in U_R^+ \cap T_i(U_R^+)$. Consider the case $j = i$. By (2.31) and the general formula

$$r_{i,+}(xx') = q_i^{\langle \gamma', \alpha_i^\vee \rangle} r_{i,+}(x)x' + x r_{i,+}(x') \qquad (x \in U^+, x' \in U_{\gamma'}^+)$$

we easily obtain

$$x \in U_\gamma^+ \cap T_i(U^+) \implies e_i x - q_i^{\langle \gamma, \alpha_i^\vee \rangle} x e_i \in U_{\gamma+\alpha_i}^+ \cap T_i(U^+).$$

Now let $x \in U_{R,\gamma}^+ \cap T_i(U_R^+)$. Define $x_k \in U_{\gamma+k\alpha_i}^+ \cap T_i(U^+)$ inductively by $x_0 = x$, $x_{k+1} = \frac{1}{[k+1]_{q_i}}(e_i x_k - q_i^{\langle \gamma, \alpha_i^\vee \rangle + 2k} x_k e_i)$. Then we see by induction on n that

$$(3.6) \qquad e_i^{(n)} x = \sum_{k=0}^n q_i^{(n-k)(\langle \gamma, \alpha_i^\vee \rangle + k)} x_k e_i^{(n-k)},$$

or equivalently,

$$(3.7) \qquad x_n = e_i^{(n)} x - \sum_{k=0}^{n-1} q_i^{(n-k)(\langle \gamma, \alpha_i^\vee \rangle + k)} x_k e_i^{(n-k)}.$$

We obtain from (3.7) that $x_n \in U_R^+$ by induction on n. By $T_i(U_R) = U_R$ we have $x_n \in U_R^+ \cap T_i(U^+) = U_R^+ \cap T_i(U_R^+)$. It follows that $e_i^{(n)}(U_R^+ \cap T_i(U_R^+)) \subset \sum_{k=0}^n (U_R^+ \cap T_i(U_R^+)) e_i^{(k)}$ by (3.6). $\qquad \square$

We set

$$\sharp U_R^0 = \bigoplus_{\gamma \in Q} R k_\gamma \subset U_R^0, \qquad \sharp U_R^{\geqq 0} = \sharp U_R^0 U_R^+, \qquad \sharp U_R^{\leqq 0} = \sharp U_R^0 U_R^-.$$

Define a subring $\tilde{\mathbb{A}}$ of $\mathbb{F}$ by

$$(3.8) \qquad \begin{aligned} \tilde{\mathbb{A}} &= \mathbb{Z}[q, q^{-1}, (q - q^{-1})^{-1}, [n]_q^{-1} \mid n > 0] \\ &= \mathbb{Z}[q, q^{-1}, (q^n - 1)^{-1} \mid n > 0]. \end{aligned}$$

Then the Drinfeld pairing induces a bilinear form

$$\tau_{\tilde{\mathbb{A}}} : {}^\sharp U_{\tilde{\mathbb{A}}}^{\geqq 0} \times {}^\sharp U_{\tilde{\mathbb{A}}}^{\leqq 0} \to \tilde{\mathbb{A}}.$$

For $\gamma \in Q^+$ we denote its restriction to $U_{\tilde{\mathbb{A}}, \gamma}^+ \times U_{\tilde{\mathbb{A}}, -\gamma}^-$ by

$$\tau_{\tilde{\mathbb{A}}, \gamma} : U_{\tilde{\mathbb{A}}, \gamma}^+ \times U_{\tilde{\mathbb{A}}, -\gamma}^- \to \tilde{\mathbb{A}}.$$

In the rest of this paper we fix a field K and $z \in K^\times$ which is not a root of 1, and consider the Hopf algebra

$$(3.9) \qquad U_z = K \otimes_{\tilde{\mathbb{A}}} U_{\tilde{\mathbb{A}}},$$

where $\tilde{\mathbb{A}} \to K$ is given by $q \mapsto z$. We define subalgebras U_z^0, U_z^+, U_z^-, $U_z^{\geqq 0}$, $U_z^{\leqq 0}$ of U_z by

$$U_z^0 = K \otimes_{\tilde{\mathbb{A}}} U_{\tilde{\mathbb{A}}}^0, \quad U_z^\pm = K \otimes_{\tilde{\mathbb{A}}} U_{\tilde{\mathbb{A}}}^\pm,$$
$$U_z^{\geqq 0} = K \otimes_{\tilde{\mathbb{A}}} U_{\tilde{\mathbb{A}}}^{\geqq 0}, \quad U_z^{\leqq 0} = K \otimes_{\tilde{\mathbb{A}}} U_{\tilde{\mathbb{A}}}^{\leqq 0}.$$

For $\gamma \in Q^+$ we set $U_{z, \pm \gamma}^\pm = K \otimes_{\tilde{\mathbb{A}}} U_{\tilde{\mathbb{A}}, \pm \gamma}^\pm$. Then we have

$$U_z^0 = \bigoplus_{h \in \mathfrak{h}_{\mathbb{Z}}} K k_h, \qquad U_z^\pm = \bigoplus_{\gamma \in Q^+} U_{z, \pm \gamma}^\pm.$$

By (3.1) we have

$$(3.10) \qquad \sum_{\gamma \in Q^+} \dim U_{z, -\gamma}^- e(-\gamma) = D^{-1}.$$

Moreover, setting

$$U_{z, \gamma} = \{u \in U_z \mid k_h u k_h^{-1} = z^{\langle \gamma, h \rangle} u \quad (h \in \mathfrak{h}_{\mathbb{Z}})\} \qquad (\gamma \in Q),$$

we have $U^{\pm}_{z,\pm\gamma} = U^{\pm}_z \cap U_{z,\gamma}$ since z is not a root of 1. The multiplication of U_z induces isomorphisms

$$(3.11) \qquad U_z \cong U^+_z \otimes U^0_z \otimes U^-_z \cong U^-_z \otimes U^0_z \otimes U^+_z,$$

$$(3.12) \quad U^{\geqq 0}_z \cong U^+_z \otimes U^0_z \cong U^0_z \otimes U^+_z, \quad U^{\leqq 0}_z \cong U^-_z \otimes U^0_z \cong U^0_z \otimes U^-_z$$

of K-modules. Here, $\otimes$ denotes $\otimes_K$.

For a U_z-module V and $\lambda \in P$ we set

$$V_\lambda = \{v \in V \mid k_h v = z^{\langle \lambda, h \rangle} v \ (h \in \mathfrak{h}_{\mathbb{Z}})\}.$$

We say that a U_z-module V is integrable if $V = \bigoplus_{\lambda \in P} V_\lambda$ and for any $v \in V$ and $i \in I$ there exists some $N > 0$ such that $e^{(n)}_i v = f^{(n)}_i v = 0$ for $n \geqq N$.

For $i \in I$ and an integrable U_z-module V define an operator $T_i : V \to V$ by

$$T_i v = \sum_{-a+b-c=\langle \lambda, h_i \rangle} (-1)^b z^{-ac+b}_i e^{(a)}_i f^{(b)}_i e^{(c)}_i v \qquad (v \in V_\lambda),$$

where $z_i = z^{(\alpha_i, \alpha_i)/2}$. It is invertible, and satisfies $T_i V_\lambda = V_{s_i \lambda}$ for $\lambda \in P$. We denote by $T_i : U_z \to U_z$ the algebra automorphism of U_z induced from $T_i : U_{\tilde{\mathbb{A}}} \to U_{\tilde{\mathbb{A}}}$. Then we have $T_i(U_{z,\gamma}) = U_{z,s_i\gamma}$ for $\gamma \in Q$.

LEMMA **3.2.** *The multiplication of U_z induces isomorphisms*

$$(3.13) \qquad U^+_z \cong (U^+_z \cap T_i(U^+_z)) \otimes \left(\bigoplus^{\infty}_{n=0} K e^{(n)}_i \right),$$

$$(3.14) \qquad U^+_z \cong \left(\bigoplus^{\infty}_{n=0} K e^{(n)}_i \right) \otimes (U^+_z \cap T^{-1}_i(U^+_z)),$$

$$(3.15) \qquad U^-_z \cong (U^-_z \cap T_i(U^-_z)) \otimes \left(\bigoplus^{\infty}_{n=0} K f^{(n)}_i \right),$$

$$(3.16) \qquad U^-_z \cong \left(\bigoplus^{\infty}_{n=0} K f^{(n)}_i \right) \otimes (U^-_z \cap T^{-1}_i(U^-_z)).$$

PROOF. We only show (3.13). By Lemma 3.1 we have

$$U^+_z \cong \left(K \otimes_{\tilde{\mathbb{A}}} (U^+_{\tilde{\mathbb{A}}} \cap T_i(U^+_{\tilde{\mathbb{A}}})) \right) \otimes \left(\bigoplus^{\infty}_{n=0} K e^{(n)}_i \right).$$

By $U^+_{\tilde{\mathbb{A}}} \cap T_i(U^+_{\tilde{\mathbb{A}}}) = U^+_{\tilde{\mathbb{A}}} \cap T_i(U^+)$ the canonical map $K \otimes_{\tilde{\mathbb{A}}} (U^+_{\tilde{\mathbb{A}}} \cap T_i(U^+_{\tilde{\mathbb{A}}})) \to U^+_z \cap T_i(U^+_z)$ is injective. Hence we have a sequence of

canonical maps

$$U_z^+ \cong \left(K \otimes_{\tilde{\mathbb{A}}} (U_{\tilde{\mathbb{A}}}^+ \cap T_i(U_{\tilde{\mathbb{A}}}^+)) \right) \otimes \left(\bigoplus_{n=0}^{\infty} K e_i^{(n)} \right)$$

$$\hookrightarrow (U_z^+ \cap T_i(U_z^+)) \otimes \left(\bigoplus_{n=0}^{\infty} K e_i^{(n)} \right) \to U_z^+.$$

Therefore, it is sufficient to show that

$$(U_z^+ \cap T_i(U_z^+)) \otimes \left(\bigoplus_{n=0}^{\infty} K e_i^{(n)} \right) \to U_z$$

is injective. This follows by applying T_i to $U_z^+ \otimes U_z^{\leqq 0} \cong U_z$. $\qquad\square$

We set

$$^{\sharp}U_z^0 = K \otimes_{\tilde{\mathbb{A}}} {}^{\sharp}U_{\tilde{\mathbb{A}}}^0, \quad {}^{\sharp}U_z^{\geqq 0} = K \otimes_{\tilde{\mathbb{A}}} {}^{\sharp}U_{\tilde{\mathbb{A}}}^{\geqq 0}, \quad {}^{\sharp}U_z^{\leqq 0} = K \otimes_{\tilde{\mathbb{A}}} {}^{\sharp}U_{\tilde{\mathbb{A}}}^{\leqq 0}.$$

They are Hopf subalgebras of U_z. The Drinfeld pairing induces a bi-linear form

$$\tau_z : {}^{\sharp}U_z^{\geqq 0} \times {}^{\sharp}U_z^{\leqq 0} \to K.$$

For $\gamma \in Q^+$ we denote its restriction to $U_{z,\gamma}^+ \times U_{z,-\gamma}^-$ by

$$\tau_{z,\gamma} : U_{z,\gamma}^+ \times U_{z,-\gamma}^- \to K.$$

4. THE MODIFIED ALGEBRA

Set

$$\begin{aligned} J_z^+ &= \{ x \in U_z^+ \mid \tau_z(x, U_z^-) = \{0\} \}, \\ J_z^- &= \{ y \in U_z^- \mid \tau_z(U_z^+, y) = \{0\} \}. \end{aligned}$$

For $\gamma \in Q^+$ we set

$$J_{z,\pm\gamma}^{\pm} = J_z^{\pm} \cap U_{z,\pm\gamma}^{\pm}.$$

By (2.26) we have

$$(4.1) \qquad\qquad J_z^{\pm} = \bigoplus_{\gamma \in Q^+ \setminus \{0\}} J_{z,\pm\gamma}^{\pm}.$$

Define a two-sided ideal J_z of U_z by

$$J_z = U_z J_z^+ U_z + U_z J_z^- U_z.$$

PROPOSITION **4.1.** (i) *We have*

$$\Delta(J_z) \subset U_z \otimes J_z + J_z \otimes U_z, \quad \varepsilon(J_z) = \{0\}, \quad S(J_z) \subset J_z.$$

 (ii) *Under the isomorphism* $U_z \cong U_z^+ \otimes U_z^0 \otimes U_z^-$ *(resp.* $U_z \cong U_z^- \otimes U_z^0 \otimes U_z^+$*) induced by the multiplication of* U_z *we have*

$$J_z \cong J_z^+ \otimes U_z^0 \otimes U_z^- + U_z^+ \otimes U_z^0 \otimes J_z^-,$$

$$(\text{resp. } J_z \cong J_z^- \otimes U_z^0 \otimes U_z^+ + U_z^- \otimes U_z^0 \otimes J_z^+).$$

PROOF. (i) It is sufficient to show

$$(4.2) \qquad \Delta(J_z^+) \subset J_z^{+\sharp}U_z^0 \otimes U_z^+ + {}^\sharp U_{\underline{z}}^{\geq 0} \otimes J_z^+,$$

$$(4.3) \qquad \Delta(J_z^-) \subset J_z^- \otimes {}^\sharp U_{\underline{z}}^{\leq 0} + {}^\sharp U_z^- \otimes J_z^{-\sharp}U_z^0,$$

$$(4.4) \qquad \varepsilon(J_z^\pm) = \{0\},$$

$$(4.5) \qquad S(J_z^\pm) \subset J_z^{\pm\sharp}U_z^0.$$

By (2.25) we have

$$J_z^{+\sharp}U_z^0 = \{x \in {}^\sharp U_{\underline{z}}^{\geq 0} \mid \tau_z(x, U_z^-) = \{0\}\}.$$

Hence in order to verify (4.2) it is sufficient to show

$$\tau_z(\Delta(J_z^+), U_z^- \otimes U_z^-) = \{0\}.$$

This follows from (2.20). The proof of (4.3) is similar. The assertions (4.4) and (4.5) follow from (4.1) and (2.28), respectively.

(ii) It is sufficient to show

$$(4.6) \qquad J_z^\pm U_z^\pm = U_z^\pm J_z^\pm = J_z^\pm,$$

$$(4.7) \qquad J_z^+ U_{\underline{z}}^{\leq 0} = U_{\underline{z}}^{\leq 0} J_z^+, \qquad J_z^- U_{\underline{z}}^{\geq 0} = U_{\underline{z}}^{\geq 0} J_z^-.$$

The assertion (4.6) follows from (2.20), (2.21), (2.25). By (4.1) we have $J_z^\pm U_z^0 = U_z^0 J_z^\pm$. Hence in order to show (4.7) it is sufficient to show $J_z^{+\sharp}U_{\underline{z}}^{\leq 0} = {}^\sharp U_{\underline{z}}^{\leq 0} J_z^+$ and $J_z^{-\sharp}U_{\underline{z}}^{\geq 0} = {}^\sharp U_{\underline{z}}^{\geq 0} J_z^-$. Let $x \in J_z^+$, $y \in {}^\sharp U_{\underline{z}}^{\leq 0}$. By (4.2) we have

$$\Delta_2(x)$$
$$\in {}^\sharp U_{\underline{z}}^{\geq 0} \otimes {}^\sharp U_{\underline{z}}^{\geq 0} \otimes J_z^+ + {}^\sharp U_{\underline{z}}^{\geq 0} \otimes J_z^{+\sharp}U_z^0 \otimes U_z^+ + J_z^{+\sharp}U_z^0 \otimes {}^\sharp U_{\underline{z}}^{\geq 0} \otimes U_z^+.$$

Hence we have $xy \in {}^\sharp U_{\underline{z}}^{\leq 0} J_z^+$ and $yx \in J_z^{+\sharp}U_{\underline{z}}^{\leq 0}$ by (2.29), (2.30). It follows that $J_z^{+\sharp}U_{\underline{z}}^{\leq 0} = {}^\sharp U_{\underline{z}}^{\leq 0} J_z^+$. The proof of $J_z^{-\sharp}U_{\underline{z}}^{\geq 0} = {}^\sharp U_{\underline{z}}^{\geq 0} J_z^-$ is similar. $\qquad \square$

By (2.35), (2.36), (2.37) we see easily the following.

LEMMA **4.2.** *For $i \in I$ we have*

$$J_z^- \cong (J_z^- \cap T_i(U_z^-)) \otimes \left(\bigoplus_{n=0}^\infty K f_i^{(n)} \right),$$

$$J_z^- \cong \left(\bigoplus_{n=0}^\infty K f_i^{(n)} \right) \otimes (J_z^- \cap T_i^{-1}(U_z^-)).$$

Moreover, we have

$$T_i^{-1}(J_z^- \cap T_i(U_z^-)) = J_z^- \cap T_i^{-1}(U_z^-).$$

We set

$$(4.8) \qquad \overline{U}_z = U_z / J_z.$$

It is a Hopf algebra by Proposition 4.1. Denote by $\overline{U}_z^0$, $\overline{U}_z^\pm$, $\overline{U}_z^{\geq 0}$, $\overline{U}_z^{\leq 0}$, $\sharp\overline{U}_z^0$, $\sharp\overline{U}_z^{\geq 0}$, $\sharp\overline{U}_z^{\leq 0}$, $\overline{U}_{z,\pm\gamma}^\pm$ $(\gamma \in Q^+)$ the images of U_z^0, $U_z^\pm$, $U_z^{\geq 0}$, $U_z^{\leq 0}$, $\sharp U_z^0$, $\sharp U_z^{\geq 0}$, $\sharp U_z^{\leq 0}$, $U_{z,\pm\gamma}^\pm$ under $U_z \to \overline{U}_z$ respectively. By the above argument we have

$$\overline{U}_z \cong \overline{U}_z^+ \otimes \overline{U}_z^0 \otimes \overline{U}_z^- \cong \overline{U}_z^- \otimes \overline{U}_z^0 \otimes \overline{U}_z^+,$$

$$\overline{U}_z^{\geq 0} \cong \overline{U}_z^+ \otimes \overline{U}_z^0 \cong \overline{U}_z^0 \otimes \overline{U}_z^+, \qquad \overline{U}_z^{\leq 0} \cong \overline{U}_z^- \otimes \overline{U}_z^0 \cong \overline{U}_z^0 \otimes \overline{U}_z^-,$$

$$\sharp\overline{U}_z^{\geq 0} \cong \overline{U}_z^+ \otimes \sharp\overline{U}_z^0 \cong \sharp\overline{U}_z^0 \otimes \overline{U}_z^+, \qquad \sharp\overline{U}_z^{\leq 0} \cong \overline{U}_z^- \otimes \sharp\overline{U}_z^0 \cong \sharp\overline{U}_z^0 \otimes \overline{U}_z^-,$$

$$\overline{U}_z^0 \cong U_z^0 = \bigoplus_{h\in\mathfrak{h}_\mathbb{Z}} K k_h, \qquad \sharp\overline{U}_z^0 \cong \sharp U_z^0 = \bigoplus_{\gamma\in Q} K k_\gamma,$$

and

(4.9) $$\overline{U}_z^\pm = \bigoplus_{\gamma\in Q^+} \overline{U}_{z,\pm\gamma}^\pm, \qquad \overline{U}_{z,\pm\gamma}^\pm \cong U_{z,\pm\gamma}^\pm / J_{z,\pm\gamma}^\pm.$$

By definition τ_z induces a bilinear form

$$\overline{\tau}_z : \sharp\overline{U}_z^{\geq 0} \times \sharp\overline{U}_z^{\leq 0} \to K$$

such that for any $\gamma \in Q^+$ its restriction

$$\overline{\tau}_{z,\gamma} : \overline{U}_{z,\gamma}^+ \times \overline{U}_{z,-\gamma}^- \to K$$

is non-degenerate.

For $\lambda \in P$ and a $\overline{U}_z$-module V we set

$$V_\lambda = \{v \in V \mid k_h v = z^{\langle\lambda,h\rangle} v \ (h \in \mathfrak{h}_\mathbb{Z})\}.$$

We define a category $\mathcal{O}(\overline{U}_z)$ as follows. Its objects are $\overline{U}_z$-modules V which satisfy

(4.10) $$V = \bigoplus_{\lambda\in P} V_\lambda, \qquad \dim V_\lambda < \infty \quad (\lambda \in P),$$

and such that there exist finitely many $\lambda_1, \ldots, \lambda_r \in P$ such that

$$\{\lambda \in P \mid V_\lambda \neq \{0\}\} \subset \bigcup_{k=1}^r (\lambda_k - Q^+).$$

The morphisms are homomorphisms of $\overline{U}_z$-modules.

We say that a $\overline{U}_z$-module V is integrable if $V = \bigoplus_{\lambda\in P} V_\lambda$ and for any $v \in V$ there exists $N > 0$ such that for $i \in I$ and $n \geqq N$ we have $e_i^{(n)} v = f_i^{(n)} v = 0$. We denote by $\mathcal{O}^{\mathrm{int}}(\overline{U}_z)$ the full subcategory of $\mathcal{O}(\overline{U}_z)$ consisting of integrable $\overline{U}_z$-modules belonging to $\mathcal{O}(\overline{U}_z)$.

For each coset $C = \mu + Q \in P/Q$ we denote by $\mathcal{O}_C(\overline{U}_z)$ the full subcategory of $\mathcal{O}(\overline{U}_z)$ consisting of $V \in \mathcal{O}_C(\overline{U}_z)$ such that $V = \bigoplus_{\lambda\in C} V_\lambda$. We also set $\mathcal{O}_C^{\mathrm{int}}(\overline{U}_z) = \mathcal{O}_C(\overline{U}_z) \cap \mathcal{O}^{\mathrm{int}}(\overline{U}_z)$. Then we have

(4.11) $$\mathcal{O}(\overline{U}_z) = \bigoplus_{C\in P/Q} \mathcal{O}_C(\overline{U}_z), \qquad \mathcal{O}^{\mathrm{int}}(\overline{U}_z) = \bigoplus_{C\in P/Q} \mathcal{O}_C^{\mathrm{int}}(\overline{U}_z).$$

For $\lambda \in P$ we define $M_z(\lambda) \in \mathcal{O}_{\lambda+Q}(\overline{U}_z)$ by

$$M_z(\lambda) = \overline{U}_z / \left(\sum_{h \in \mathfrak{h}_{\mathbb{Z}}} \overline{U}_z(k_h - z^{\langle \lambda, h \rangle}) + \sum_{i \in I} \overline{U}_z e_i \right),$$

and for $\lambda \in P^+$ we define $V_z(\lambda) \in \mathcal{O}_{\lambda+Q}^{\mathrm{int}}(\overline{U}_z)$ by

$$V_z(\lambda) = \overline{U}_z / \left(\sum_{h \in \mathfrak{h}_{\mathbb{Z}}} \overline{U}_z(k_h - z^{\langle \lambda, h \rangle}) + \sum_{i \in I} \overline{U}_z e_i + \sum_{i \in I} \overline{U}_z f_i^{(\langle \lambda, h_i \rangle + 1)} \right).$$

Let $\lambda \in P$. A $\overline{U}_z$-module V is called a highest weight module with highest weight λ if there exists $v \in V_\lambda \setminus \{0\}$ such that $V = \overline{U}_z v$ and $xv = \varepsilon(x)v$ $(x \in \overline{U}_z^+)$. Then we have $V \in \mathcal{O}_{\lambda+Q}(\overline{U}_z)$. A $\overline{U}_z$-module is a highest weight module with highest weight λ if and only if it is a non-zero quotient of $M_z(\lambda)$. If there exists an integrable highest weight module with highest weight λ, then we have $\lambda \in P^+$. For $\lambda \in P^+$ a $\overline{U}_z$-module is an integrable highest weight module with highest weight λ if and only if it is a non-zero quotient of $V_z(\lambda)$.

For $V \in \mathcal{O}(\overline{U}_z)$ we define its formal character by

$$\mathrm{ch}(V) = \sum_{\lambda \in P} \dim V_\lambda e(\lambda) \in \mathcal{E}.$$

We have

$$\mathrm{ch}(M_z(\lambda)) = e(\lambda)\overline{D}^{-1} \qquad (\lambda \in P),$$

where

$$\overline{D}^{-1} = \sum_{\gamma \in Q^+} \dim \overline{U}_{z,-\gamma}^- e(-\gamma) \qquad (\lambda \in P).$$

For each coset $C = \mu + Q \in P/Q$ we fix a function $f_C : C \to \mathbb{Z}$ such that

$$f_C(\lambda) - f_C(\lambda - \alpha_i) = 2\langle \lambda, t_i \rangle \qquad (\lambda \in C, i \in I).$$

REMARK **4.3.** The function f_C is unique up to addition of a constant function. If we extend $(\ ,\ ) : E \times E \to \mathbb{Q}$ to a W-invariant symmetric bilinear form on $\mathfrak{h}^*$, then f_C is given by

$$f_C(\lambda) = (\lambda + \rho, \lambda + \rho) + a \qquad (\lambda \in C)$$

for some $a \in \mathbb{Q}$.

For $\gamma \in Q^+$ let $\overline{C}_\gamma \in \overline{U}_{z,\gamma}^+ \otimes \overline{U}_{z,-\gamma}^-$ be the canonical element of the non-degenerate bilinear form $\overline{\tau}_{z,\gamma}$. Following Drinfeld we set

$$\Omega_\gamma = (m \circ (S \otimes 1) \circ P)(\overline{C}_\gamma) \in \overline{U}_{z,-\gamma}^- \overline{U}_z^0 \overline{U}_{z,\gamma}^+,$$

where $m : \overline{U}_z \otimes \overline{U}_z \to \overline{U}_z$ and $P : \overline{U}_z \otimes \overline{U}_z \to \overline{U}_z \otimes \overline{U}_z$ are given by $m(a,b) = ab$, $P(a \otimes b) = b \otimes a$ (see [12, Section 3.2], [11, Section 6.1]). Let $C \in P/Q$. For $V \in \mathcal{O}_C(\overline{U}_z)$ we define a linear map

$$(4.12) \qquad\qquad\qquad \Omega : V \to V$$

by

$$\Omega(v) = z^{f_C(\lambda)} \sum_{\gamma \in Q^+} \Omega_\gamma v \qquad (v \in V_\lambda).$$

This operator is called the quantum Casimir operator. As in [12, Section 3.2] we have the following.

PROPOSITION **4.4.** *Let* $C \in P/Q$. *For* $\lambda \in C$ *the operator* Ω *acts on* $M_z(\lambda)$ *as* $z^{f_C(\lambda)}$ id.

Since z is not a root of 1, we have

$$z^{f_C(\lambda)} = z^{f_C(\mu)} \implies f_C(\lambda) = f_C(\mu).$$

5. MAIN RESULTS

For $w \in W$ and $x = \sum_{\lambda \in P} c_\lambda e(\lambda) \in \mathcal{E}$ we set

$$wx = \sum_{\lambda \in P} c_\lambda e(w\lambda), \qquad w \circ x = \sum_{\lambda \in P} c_\lambda e(w \circ \lambda).$$

The elements wx, $w \circ x$ may not belong to $\mathcal{E}$; however, we will only consider the case where $wx, w \circ x \in \mathcal{E}$.

We denote by sgn $: W \to \{\pm 1\}$ the character given by $\mathrm{sgn}(s_i) = -1$ for $i \in I$.

PROPOSITION **5.1.** *For any* $w \in W$ *we have* $w \circ \overline{D} = \mathrm{sgn}(w)\overline{D}$.

PROOF. We may assume that $w = s_i$ for $i \in I$. Define $D_i, \overline{D}_i \in \mathcal{E}$ by

$$D = (1 - e(-\alpha_i))D_i, \qquad \overline{D} = (1 - e(-\alpha_i))\overline{D}_i.$$

Then we have $D_i = \prod_{\alpha \in \Delta^+ \setminus \{\alpha_i\}} (1 - e(-\alpha))^{m_\alpha}$. Moreover, by Lemma 3.2, Lemma 4.2 and (4.9) we have

$$\begin{aligned}
D_i^{-1} &= \sum_{\gamma \in Q^+} \dim(U_{z,-\gamma}^- \cap T_i(U_z^-))e(-\gamma) \\
&= \sum_{\gamma \in Q^+} \dim(U_{z,-\gamma}^- \cap T_i^{-1}(U_z^-))e(-\gamma), \\
\overline{D}_i^{-1} &= D_i^{-1} - \sum_{\gamma \in Q^+} \dim(J_{z,-\gamma}^- \cap T_i(U_z^-))e(-\gamma) \\
&= D_i^{-1} - \sum_{\gamma \in Q^+} \dim(J_{z,-\gamma}^- \cap T_i^{-1}(U_z^-))e(-\gamma).
\end{aligned}$$

By $s_i \circ \overline{D} = -(1 - e(-\alpha_i))s_i\overline{D}_i$ we have only to show $s_i\overline{D}_i = \overline{D}_i$. By Lemma 4.2 we have

$$s_i\big(\sum_{\gamma \in Q^+} \dim(J_{z,-\gamma}^- \cap T_i(U_z^-))e(-\gamma)\big)$$

$$= \sum_{\gamma \in Q^+} \dim(J_z^- \cap T_i(U_{z,-\gamma}^-))e(-\gamma)$$

$$= \sum_{\gamma \in Q^+} \dim(J_{z,-\gamma}^- \cap T_i^{-1}(U_z^-))e(-\gamma),$$

and hence the assertion follows from $s_iD_i = D_i$. $\qquad\qquad\square$

PROPOSITION **5.2.** *Let* $\lambda \in P^+$. *Assume that* V *is an integrable highest weight* $\overline{U}_z$-*module with highest weight* λ. *Then we have*

$$\mathrm{ch}(V) = \sum_{w \in W} \mathrm{sgn}(w)\,\mathrm{ch}(M_z(w \circ \lambda)).$$

PROOF. The proof below is the same as the one for Lie algebras in Kac [6, Theorem 10.4].

Set $C = \lambda + Q \in P/Q$. Similarly to [6, Proposition 9.8] we have

$$(5.1) \quad \mathrm{ch}(V) = \sum_{\mu \in \lambda - Q^+, f_C(\mu) = f_C(\lambda)} c_\mu\,\mathrm{ch}(M_z(\mu)) \qquad (c_\mu \in \mathbb{Z}, c_\lambda = 1).$$

Multiplying (5.1) by $\overline{D}$ we obtain

$$\overline{D}\,\mathrm{ch}(V) = \sum_{\mu \in \lambda - Q^+, f_C(\mu) = f_C(\lambda)} c_\mu e(\mu).$$

Using the action of T_i $(i \in I)$ on V we see that $w\,\mathrm{ch}(V) = \mathrm{ch}(V)$ for $w \in W$, and hence $w \circ (\overline{D}\,\mathrm{ch}(V)) = \mathrm{sgn}(w)\overline{D}\,\mathrm{ch}(V)$ for any $w \in W$. It follows that

$$(5.2) \qquad c_\mu = \mathrm{sgn}(w)c_{w \circ \mu} \qquad (\mu \in \lambda - Q^+, w \in W).$$

Assume that $\mu \in \lambda - Q^+$ satisfies $c_\mu \neq 0$. By (5.2) $W \circ \mu \subset \lambda - Q^+$, and hence we can take $\mu' \in W \circ \mu$ such that $\mathrm{ht}(\lambda - \mu')$ is minimal, where $\mathrm{ht}(\sum_i m_i\alpha_i) = \sum_i m_i$. Then we have $\langle \mu', h_i \rangle \geqq 0$ for any $i \in I$ by $s_i \circ \mu' = \mu' - (\langle \mu', h_i \rangle + 1)\alpha_i$ and (5.2). Namely, we have $\mu' \in P^+$. Then by [6, Lemma 10.3] we obtain $\mu' = \lambda$. $\qquad\square$

REMARK **5.3.** I. Heckenberger pointed out to me that Proposition 5.2 also follows from the existence of the BGG resolution of integrable highest weight modules of quantized enveloping algebras given in [5]

Recall that any integrable highest weight module V with highest weight λ is a quotient of $V_z(\lambda)$. Proposition 5.2 tells us that its character $\mathrm{ch}(V)$ only depends on λ. It follows that any integrable highest weight module with highest weight λ is isomorphic to $V_z(\lambda)$.

Consider the case $\lambda = 0$. Since $V_z(0)$ is the trivial one-dimensional module, we obtain the identity

$$1 = \left(\sum_{w \in W} \operatorname{sgn}(w) e(w \circ 0) \right) \left(\sum_{\gamma \in Q^+} \dim \overline{U}_{z,-\gamma}^- e(-\gamma) \right)$$

in $\mathcal{E}$ by Proposition 5.2. On the other hand by the corresponding result for the Kac-Moody Lie algebra we have

$$1 = \left(\sum_{w \in W} \operatorname{sgn}(w) e(w \circ 0) \right) \left(\sum_{\gamma \in Q^+} \dim U_{z,-\gamma}^- e(-\gamma) \right).$$

It follows that $U_{z,-\gamma}^- \cong \overline{U}_{z,-\gamma}^-$ for any $\gamma \in Q^+$. By $\dim U_{z,-\gamma}^- = \dim U_{z,\gamma}^+$ and the non-degeneracy of $\overline{\tau}_{z,\gamma}$ we also have $U_{z,\gamma}^+ \cong \overline{U}_{z,\gamma}^+$ for any $\gamma \in Q^+$. We have obtained the following results.

THEOREM **5.4.** *The Drinfeld pairing*

$$\tau_{z,\gamma} : U_{z,\gamma}^+ \times U_{z,-\gamma}^- \to K$$

is non-degenerate for any $\gamma \in Q^+$.

THEOREM **5.5.** *Let $\lambda \in P^+$. Assume that V is an integrable highest weight U_z-module with highest weight λ. Then we have*

$$\operatorname{ch}(V) = D^{-1} \sum_{w \in W} \operatorname{sgn}(w) e(w \circ \lambda).$$

By Theorem 5.4 we can define the quantum Casimir operator Ω for U_z. As in [11, Section 6.2] we have the following.

THEOREM **5.6.** *Any object of $\mathcal{O}^{\mathrm{int}}(U_z)$ is a direct sum of $V_z(\lambda)$'s for $\lambda \in P^+$.*

By Theorem 5.4 we have the following.

THEOREM **5.7.** *Let $\gamma \in Q^+$. Take bases $\{x_r\}$ and $\{y_s\}$ of $U_{\mathbb{A},\gamma}^+$ and $U_{\mathbb{A},-\gamma}^-$ respectively, and set $f_\gamma = \det(\tau_{\tilde{\mathbb{A}},\gamma}(x_r, y_s))_{r,s}$. Then we have $f_\gamma \in \tilde{\mathbb{A}}^\times$. Namely, we have*

$$f_\gamma = \pm q^a f_1^{\pm 1} \cdots f_N^{\pm 1},$$

where $a \in \mathbb{Z}$, and $f_1, \ldots, f_N \in \mathbb{Z}[q]$ are cyclotomic polynomials.

PROOF. We can write $f_\gamma = mgh$, where $m \in \mathbb{Z}_{>0}$, $g \in \mathbb{Z}[q]$ is a primitive polynomial with $g(0) > 0$ whose irreducible factor is not cyclotomic, and $h \in \tilde{\mathbb{A}}^\times$. Note that for any field K and $z \in K^\times$ which is not a root of 1, the specialization of f_γ with respect to the ring homomorphism $\tilde{\mathbb{A}} \to K$ ($q \mapsto z$) is non-zero by Theorem 5.4. Hence we see easily that $m = 1$ and $g = 1$. $\qquad \square$

In the finite case Theorem 5.7 is well-known (see [8], [9], [11]). In the affine case this is a consequence of Damiani [3], [4], where $\det(\tau_{\tilde{\mathbb{A}},\gamma}(x_r, y_s))_{r,s}$ is determined explicitly by a case-by-case calculation.

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